Resolution of the Canonical Fiber Metrics for a Lefschetz Fibration

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Joint work with Richard Melrose

Constant scalar curvature metrics

Uniformization theorem

Every Riemann surface with genus > 1 admits a metric with constant scalar curvature -1.

What if the surface becomes singular?

Question

(a) Does there exist a constant scalar curvature metric on the singular surface?
(b) If yes, how does it evolve when approaching the singular geometry?

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Singular geometry

Take a nontrivial geodesic cycle in M, and let its length go to zero.

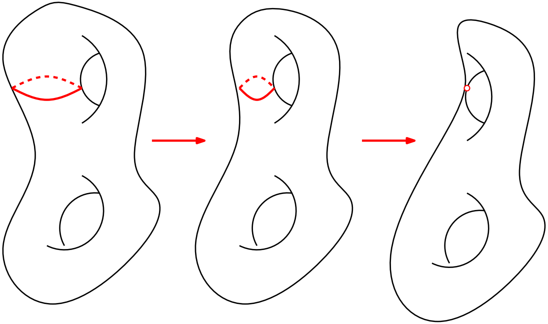

Figure: Degenerating surfaces with a geodesic cycle shrinking to a point

Singular geometry

Take a nontrivial geodesic cycle in M, and let its length go to zero. This process can be indexed by a complex parameter $t \in \mathbb{D}$.

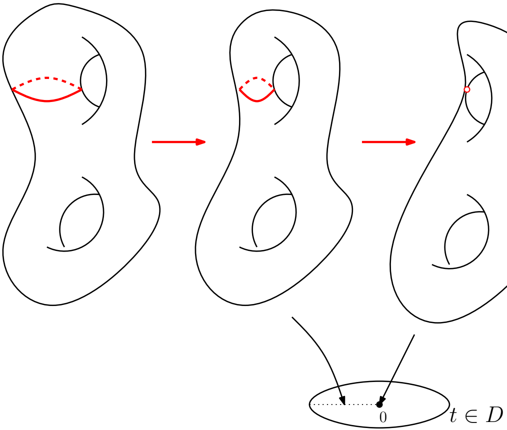

Figure: Degenerating surfaces, indexed by a parameter t

Local geometry: hyperbolic cylinder

Locally the geometry near the shrinking cycle is described by the normal crossing model:

$$(z, w) \in \mathbb{C}^2, \quad zw = t$$

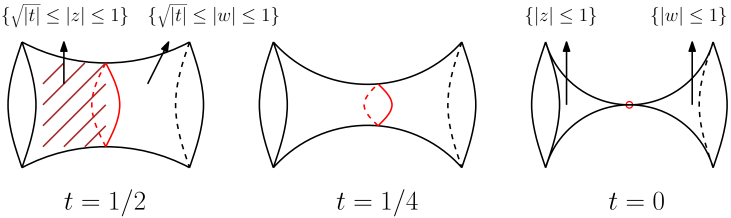

Figure: Local geometry of $zw = t$, with coordinate patch z and w

Lefschetz fibration

This local behavior fits naturally with Lefschetz fibration.

Definition

For a compact connected almost-complex 4-manifold M, and a Riemann surface Z, a Lefschetz fibration is defined as
$\psi : M^4 \to Z^2$

- has regular complex fibers except one point $p \in Z$;
- has surjective differential outside one point $q \in \psi^{-1}(p)$;
- near q is reducible to the model $\{zw = t\}$.

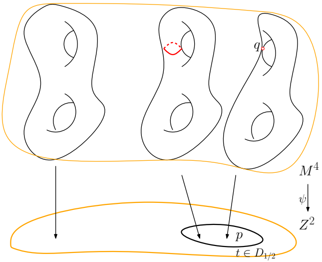

Figure: Lefschetz fibration
$\psi : M \to Z$

Geometric background of Lefschetz fibration

Lefschetz fibration is an interesting geometric object:

- Any algebraic surface is birational to a surface with Lefschetz fibration.
- A four-dimensional simply-connected compact symplectic manifold, possibly after stabilization by a finite number of blow-ups, admits a Lefschetz fibration over the sphere [Donaldson, 1998]
- The converse is also true: if the homology class of the regular fibre $[F]$ is nonzero in $H_2(X;\mathbb{R})$, then X has to be symplectic. [Gompf, 1999]

D–M compactification of moduli space

Lefschetz fibration also appears to be the extremal behavior in Deligne–Mumford compactification.

- Space of universal curves fiber over the moduli space $\mathcal{C}_{g,p} \longrightarrow \mathcal{M}_{g,p}$.
- For $g > 1$, Deligne–Mumford compactification $\overline{\mathcal{C}_{g,p}}$ adds exceptional divisors, appearance of pairs of nodal points on a Riemann surface of lower genus. [Deligne–Mumford, 1979]
- Metrics on the universal curves provide information on Weil–Petersson metric on moduli space: [Masur, 1976] [Masur–Wolf, 2002] [Mirzakhani, 2007] [Mazzeo–Swoboda, ongoing]

D–M compactification

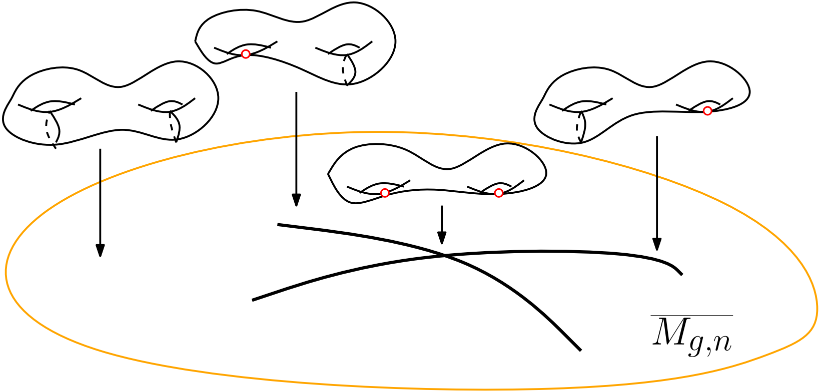

Figure: Universal curves over D–M compactified moduli space

Main theorem

Theorem[Melrose–Z, 2014]

There is a resolved space $M_{\mathrm{met}} \longrightarrow M$ such that for the plumbing metric $g_{pl}^{(t)} \in \mathrm{Met}(M_{\mathrm{met}}^{(t)})$, there exists a polyhomogeneous function $f \in (\frac{1}{\log|t|})^2 \mathcal{C}^{\infty}_{\log}(M_{\mathrm{met}})$ such that the fiber metric

$$g_{cc}^{(t)} = e^{2f} g_{pl}^{(t)}$$

satisfies

$$R(g_{cc}^{(t)}) = -1, \forall |t| \in [0, \frac{1}{2}).$$

Expansion up to 3rd order

Theorem[Obitsu–Wolpert, 2009]

Let ds^2_{cc} be the hyperbolic metric on the degenerated family R_t with m vanishing cycles, Δ the associated Laplacian, and ds_{pl} the plumbing metric that comes from gluing $ds^2_{P_t}$ with the regular part, then the metric has the following expansion

$$ds^2_{cc} = ds^2_{pl}\Big(1 - \frac{\pi^2}{3}\sum_{j=1}^{m}(\log|t_j|)^{-2}(\Delta - 2)^{-1}(\Lambda(z_j) + \Lambda(w_j)) + O\big(\sum(\log|t_j|)^{-4}\big)\Big)$$

where the function Λ is given by $\Lambda(z_j) = (s_z^4 \chi_{\psi^{-1}\mathbb{D}_{1/2}})_{s_z}, \quad s_z = \log|z_j|$.

Plumbing metric

- Consider the local model:

$$P = \{(z, w, t) \in \mathbb{C}^3; zw = t,\ |z| \leq 1,\ |w| \leq 1,\ |t| \leq 1/2\}$$
$$\longrightarrow \mathbb{D}_{\frac{1}{2}} = \{t \in \mathbb{C}; |t| \leq 1/2\}.$$

Plumbing metric on each fiber

$$g_{pl}^{(t)} = \left(\frac{\pi \log|z|}{\log|t|} \csc \frac{\pi \log|z|}{\log|t|}\right)^2 g_0,$$
$$g_0 = \left(\frac{|dz|}{|z|\log|z|}\right)^2$$

- $g_{pl}^{(t)} \to g_0$ as $t \to 0$.
- Symmetric with the change of $w = t/z$.
- Fiber curvature $= -1$.

Step 1: resolving the complex structure

To make g_{pl} smooth at $t = 0$, we need to first blow up the intersection:

$$P_{\bar{\partial}} := [P; z = w = 0].$$

$$P_{\bar{\partial}} = \{(r_z, r_w); 0 \leq r_z, r_w \leq 1, \ r_z r_w \leq \frac{1}{2}\} \times \mathbb{S}_z \times \mathbb{S}_w.$$

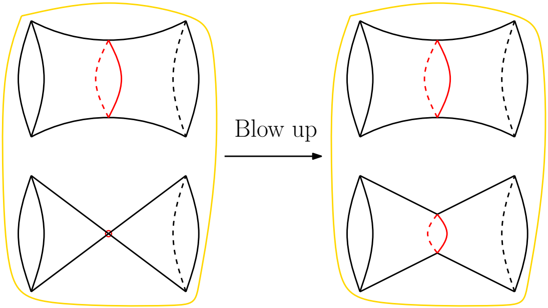

Figure: Blow up of $\{z = w = 0\}$

Step 2: Logarithmic blow up

We do a "logarithmic blow up" to the space obtained above:

$$[P; \{z = 0\}_{\log} \cup \{w = 0\}_{\log}].$$

This step introduces smooth functions $1/\log|z|$ and $1/\log|w|$:

$$s_z = \frac{1}{\log\frac{1}{r_z}},\ s_w = \frac{1}{\log\frac{1}{r_w}}$$

Step 3

After change of variable, the metric becomes

$$g_{pl}^{(t)} = \frac{\pi^2 s_t^2}{\sin^2(\frac{\pi s_t}{s_w})} \left(\frac{ds_w^2}{s_w^4} + d\theta_w^2 \right)$$

where

$$s_t = \frac{s_z s_w}{s_z + s_w} = \frac{s_w}{1 + \frac{s_w}{s_z}}$$

is not a smooth function.

We blow up, radially, the corner formed by the intersection of the two logarithmic boundary faces

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Resolved space M_{met}

We consider the following glued space of $M_{met} = (M \setminus P) \cup P_{\text{met}}$:

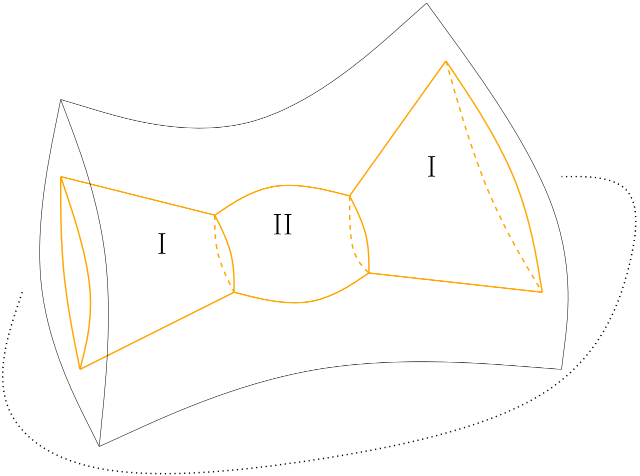

Figure: Final resolved space M_{met}

Solving the curvature equation

- We consider the following fiber metric on the space of M_{met}:

$$g_{pl} = \chi_1 g_{pl}^{(t)} + \chi_2 g_{uni}$$

χ_1: cutoff for P_{met}, $g_{pl}^{(t)}$ being the local plumbing metric on the plumbing variety
χ_2: cutoff for $M \setminus P_{\text{met}}$, with g_{uni} being the metric on a regular fiber given by the classical uniformization theorem.

- The curvature of g_{pl} satisfies

$$R(g_{pl}) = -1 + e$$

with error e is: (a) compactly supported (b) near the singular face has $O(\rho_t^2)$ decay.

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Curvature equation on M_{met}

Curvature equation for conformal factor: if $g = e^{2f} g_0$, then

$$R(g)e^{2f} = \Delta_{g_0} f + R(g_0),$$

which in our case is

$$\Delta_{g_{pl}} f + R(g_{pl}) = -e^{2f}.$$

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Solvability of $\Delta + 2$ on M_{met}

We solve the linearized equation

$$(\Delta + 2)u = f \in O(\rho_t^2)$$

on the space M_{met}.

- Two boundary faces: face I is the regular Riemann surface and face II is the one introduced in the last step
- Indicial roots: $\{1, -2\}$ for face I, and $\{-1, 2\}$ for face II
- Invertibility of $\Delta + 2$ on both surfaces
- Appearance of extra log terms

Log-smoothness of a genuine solution

Solve iteratively to get a formal expansion for the curvature equation

$$\Delta_{g_{pl}} f + R(g_{pl}) = -e^{2f}$$

where f has the following expansion

$$f \sim \sum_{k \geq 1}^{\infty} g_k$$

- g_k has a factor of ρ_t^{2k};
- Generally with logarithmics factors.

Then we use a perturbation argument to show the existence of a genuine solution.

Application: Deligne–Mumford compactification

Conjecture

There is a (non-holomorphic) resolution given below

$$\begin{array}{ccc} \widehat{\mathcal{C}_{g,p}} & \longrightarrow & \widehat{\mathcal{M}_{g,p}} \\ \big\downarrow \beta & & \big\downarrow \beta \\ \overline{\mathcal{C}_{g,p}} & \longrightarrow & \overline{\mathcal{M}_{g,p}} \end{array}$$

which, in the category of real manifolds with corners, resolves this fiber metric that we have a complete expansion.

Constant scalar curvature Kähler metric

Theorem [Fine, 2004]

If X is a compact connected complex surface admitting a holomorphic submersion onto a complex curve Σ with fibres of genus ≥ 2, then, for all large r, the Kähler class $k_r = -c_1(V) - r * c_1(\Sigma)$ contains a constant scalar curvature Kähler metric.

- Key construction: an adiabatic limit with the constant scalar curvature metric on the fibres of X and a large multiple of a metric on the base.
- The linearized operator acting on Kähler potential: $\Delta^2 - R\Delta$

Question

Generalize to complex surfaces with a Lefschetz fibration, can we find a local adiabatic model?

Thank you for your attention!