The compactified configuration space and regularity for conical metrics

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Outline

Constant curvature metric with conical singularities

Consider a compact Riemann surface M, with the following data:

- k distinct points $\mathfrak{p} = (p_1, \ldots, p_k)$
- Angle data $\vec{\beta} = (\beta_1, \ldots, \beta_k) \in (0, \infty)^k$
- Curvature constant $K \in \{-1, 0, 1\}$
- Area A
- Conformal structure $\mathfrak{c}$ given by M

A constant curvature metric with prescribed conical singularities is a smooth metric with constant curvature, except near p_j the metric is asymptotic to a cone with angle $2\pi\beta_j$.

$$\text{(Gauss–Bonnet)} \qquad \chi(M, \vec{\beta}) := \chi(M) + \sum_{j=1}^{k} (\beta_j - 1) = \frac{1}{2\pi} KA$$

Local structure near a cone point

Locally near a cone point with angle $2\pi\beta$, there are coordinates (geodesic polar coordinates) such that the metric is given by

$$g = \begin{cases} dr^2 + \beta^2 r^2 d\theta^2 & K = 0; \\ dr^2 + \beta^2 \sin^2 r d\theta^2 & K = 1; \\ dr^2 + \beta^2 \sinh^2 r d\theta^2 & K = -1 \end{cases}$$

In the flat case, relating to the conformal structure

$$r = \frac{1}{\beta}|z|^\beta, \quad \text{then} \quad g = |z|^{2(\beta-1)}(d|z|^2 + |z|^2 d\theta^2) = |z|^{2(\beta-1)}|dz|^2$$

Some examples of constant curvature conical metrics

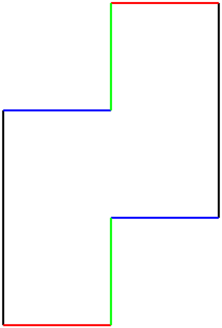


Figure: Translation surfaces

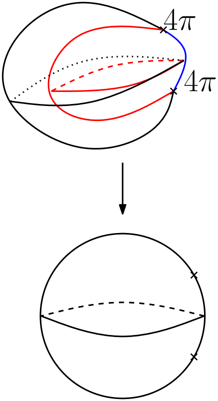


Figure: Ramified covers of constant curvature surfaces

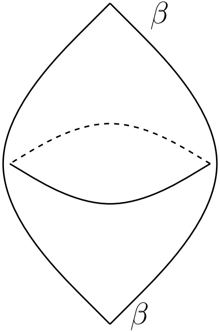


Figure: Spherical footballs

Existence and Uniqueness

Theorem (88' McOwen, 91' Troyanov, 92' Luo–Tian)

For any compact surface M and conical data $(\mathfrak{p}, \vec{\beta})$ satisfying one of the following constraints:

- $\chi(M, \vec{\beta}) \leq 0$; *or*
- $\chi(M, \vec{\beta}) > 0, \vec{\beta} \in (0, 1)^k$
    - $k = 2,\ \beta_1 = \beta_2$; *or*
    - $k \geq 3,\ \beta_j + k - \chi(M) > \sum_{i \neq j} \beta_i, \forall j.$

there is a unique constant curvature metric with the prescribed singularities.

The moduli space for $\vec{\beta} \in (0,1)^k$

Theorem (Mazzeo–Weiss, 2015)

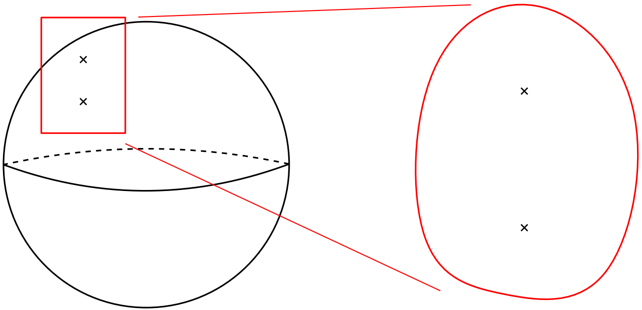
- *The spaces of constant curvature conical metrics* $\mathcal{CM}_{cc}(M, \mathfrak{p})$ *are Banach manifolds.*
- *There is an embedded* $(6\gamma - 6 + 3k)$*-dimensional submanifold* $S \subset \mathcal{CM}_{cc}(M, \mathfrak{p})$ *which is the quotient by the action of diffeomorphism group, i.e. the moduli space.*

Question

1. What happens when cone points collide?
2. Compactification of the moduli space?

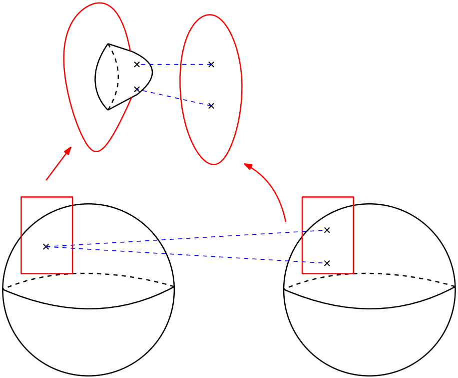
When two points collide

- Scale back the distance between two cone points ("blow up")



When two points collide

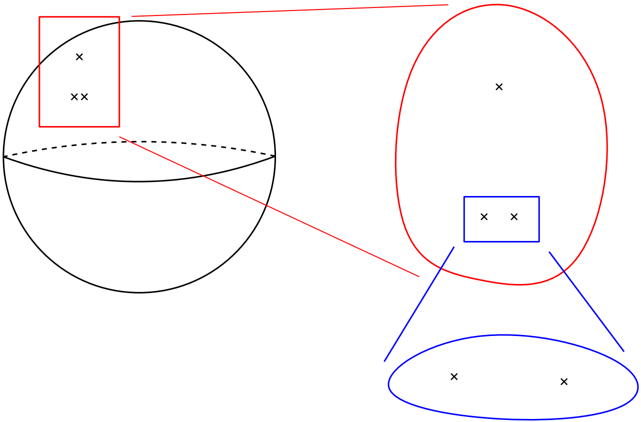
- Scale back the distance between two cone points (“blow up”)
- Half sphere at the collision point, with two cone points over the half sphere:



- Flat metric on the half sphere, and curvature K metric on the original surface

Iterative structure

- When there are several levels of distance: scale iteratively



Iterative structure

- "bubble over bubble" structure
- Higher codimensional faces from deeper scaling
- Flat conical metrics on all the new faces

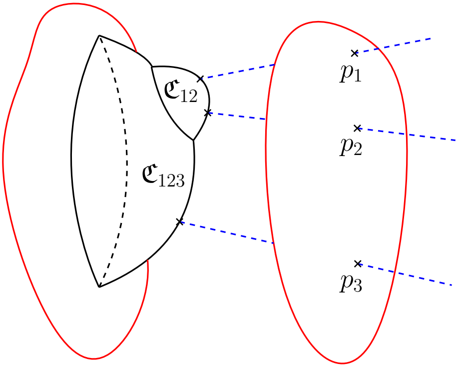


Figure: One of the singular fibers in $\mathcal{C}_3$, where two of the points collide faster than the third one

Resolution of the configuration space

This “bubbling” process can be expressed in terms of blow-up of product $M^k \times M \to M^k$

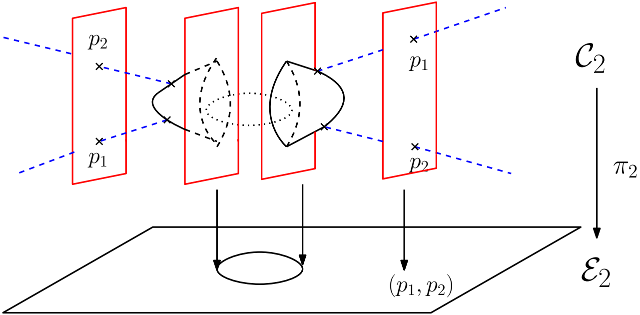


Figure: “Centered” projection of $\mathcal{C}_2 \to \mathcal{E}_2$

When there are more cone points

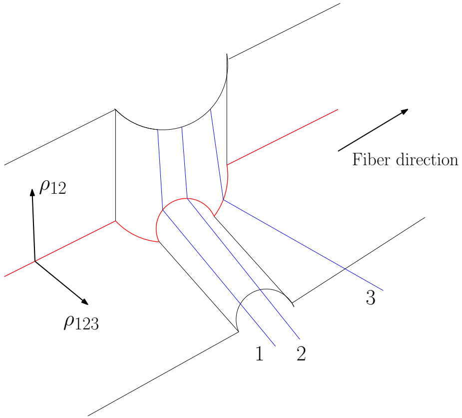


Figure: A schematic picture of resolved total space: cone points are marked blue, they become “separated” on the new faces; the most singular fiber is marked by red

Results about fiber metrics on $\mathcal{C}_k$

Theorem (Mazzeo–Z, 2017)

For any given $\vec{\beta}$, the family of constant curvature metrics with conic singularities is polyhomogeneous on $\mathcal{C}_k$.*

- *The metric family can be hyperbolic / flat (with any cone angles), or spherical (with angles less than 2π, except footballs)
- Start by constructing a model metric g_0, then solve the curvature equation uniformly

$$\Delta_{g_0} u - Ke^{2u} + K_{g_0} = 0$$

- When $K = 0$, the curvature equation is linear

The flat case

Theorem (Flat case)

The fiber flat conical metrics with fixed cone angles $\vec{\beta}$ and varying cone points $\mathfrak{p}$ lift to be polyhomogeneous on $\mathcal{C}_k$.

- The conformal factor is the sum of Green's functions
- Prove regularity by direct computation
- The metric on the front face is a rescaled conical metric with scattering (Euclidean) boundary
- Induction on the depth of the corner

Proof sketch for nonflat case

1) Construct an approximate solution, which involves iteratively solving equations on faces with increasing depth

$$\Delta_{g_0} u_N - K e^{2u_N} + K_{g_0} = \mathcal{O}(\rho^N)$$

2) Implicit function theorem to obtain the actual solution

$$\Delta_{g_0}(u_N + v_N) - K e^{2(u_N+v_N)} + K_{g_0} = 0$$

3) Commutator argument to show the regularity of v_N near the boundary faces
4) Together with the arbitrarily high order expansion, we obtain polyhomogeneity of $u = u_N + v_N$

Consequences of the theorem

Theorem

For fixed angles $\vec{\beta}$, the fiberwise hyperbolic (resp. spherical) conical metrics are polyhomogeneous on $\mathcal{C}_k$.

- The leading term of the metric is given by the flat metric
- When $K = \pm 1$, the difference from the flat metric is bounded by $O(\rho^\epsilon)$
- This matches the blow up limit

Motivation: positive curvature with big cone angles

- Literature: [Troyanov, 1991] [Umehara–Yamada, 2000] [Eremenko, 2000] [Eremenko–Gabrielov–Tarasov, 2014] [Eremenko–Gabrielov, 2015] [Bartolucci–De Marchis–Malchiodi, 2011] [Carlotto–Malchiodi, 2012] [Malchiodi, 2016]
- [Mondello–Panov, 2016]: spherical conical metrics on $\mathbb{S}^2$ under the angle "holonomy condition"

$$d_{\ell^1}(\vec{\beta} - \vec{1}, \mathbb{Z}^k_{odd}) \geq 1$$

- When the above equality holds: [Dey, 2017] [Kapovich, 2017] [Eremenko, 2017]

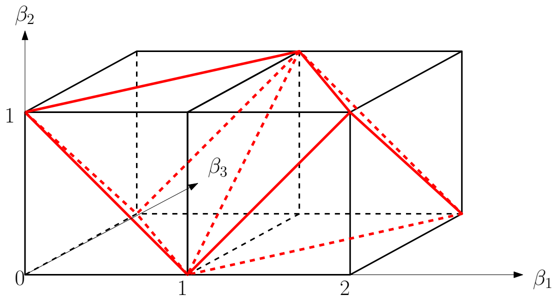


Figure: The admissible region for angle $(\beta_1, \beta_2, \beta_3)$: interior of the tetrahedra, extended by reflection symmetry

Questions to answer

Goal: Find out the structure of the moduli space

- The full solution space: for given admissible $(\vec{\beta}, \mathfrak{p}, \mathfrak{c})$, how many solutions are there?
- Deformation theory: is there a manifold structure?

Invertibility of the linearized operator $\Delta_g - 2K$

- The linearized operator for the nonlinear curvature equation is given by $\Delta_g - 2K$
- The Friedrichs extension of the Laplacian Δ_g is self-adjoint and has discrete spectrum
- When $K < 0$, $\Delta_g - 2K$ is invertible; $K = 0$, only kernel is the constant
- When $K > 0$ and $\vec{\beta} \in (0, 1)^k$, the first nonzero eigenvalue of Δ_g satisfies $\lambda_1 \geq 2K$, and equality is only achieved by the footballs
- This argument only works for all $\beta_i < 1$
- Eigenfunctions become too singular when cone angle increases, so the Lichnerowicz type argument would not work

Eigenvalue 2: obstruction of operator invertibility

- Intuition: when angles increase, eigenvalues of the Laplacian decrease
- Example: two footballs glued together

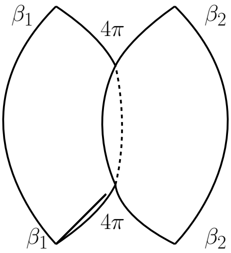


Figure: A surface with six conical points, with eigenvalue 2

- Expect stratum with eigenvalue 2 to appear in the interior, and extend to infinity.

Indicial roots

- The indicial roots of the flat conical Laplacian $r^{-2}\left((r\partial_r)^2+\beta^{-2}\partial_\theta^2\right)$
- For k-th mode, the indicial root given by

$$\pm\frac{k}{\beta},\quad \text{with kernel } r^{\pm\frac{k}{\beta}}e^{ik\theta}$$

- When Δ has eigenvalue 2, the kernel of $\Delta-2$ is prescribed by those indicial roots locally. When $\beta>1$, these roots between $(-1,0)$ are obstructions to surjectivity.

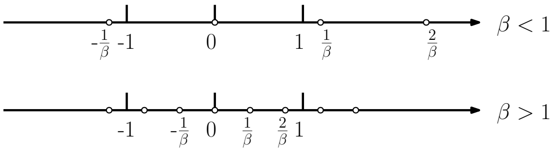


Figure: Indicial roots for different β

Geometric realization of the indicial roots

We discover that one key step to make it unobstructed is the following:

Proposition (Mazzeo–Z, in progress)

The linear space generated by the splitting of cone angles are spanned by $\{r^{-\frac{k_i}{\beta}}, 1 \leq k_i < \beta\}$.

- Proof by computing the Jacobi field generated by the geometric motion
- The linearized operator is surjective after adding those parameters
- It provides additional coordinates for the moduli space

Thank you for your attention!